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2008 J. Phys. A: Math. Theor. 41 275210

(http://iopscience.iop.org/1751-8121/41/27/275210)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 275210 (21pp) doi:10.1088/1751-8113/41/27/275210

On the global dynamics of the Rabinovich system

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Received 17 March 2008
Published 16 June 2008
Online at stacks.iop.org/JPhysA/41/275210

Abstract
In this paper by using the Poincaré compactification in $\mathbb{R}^3$ we make a global analysis of the Rabinovich system
$\dot{x} = hy - v_1x + yz, \qquad \dot{y} = hx - v_2y - xz, \qquad \dot{z} = -v_3z + xy,$
with $(x, y, z) \in \mathbb{R}^3$ and $(h, v_1, v_2, v_3) \in \mathbb{R}^4$. We give the complete description of its dynamics on the sphere at infinity. For ten sets of the parameter values the system has either first integrals or invariants. For these ten sets we provide the global phase portrait of the Rabinovich system in the Poincaré ball (i.e. in the compactification of $\mathbb{R}^3$ with the sphere $\mathbb{S}^2$ of the infinity). We prove that for convenient values of the parameters the system has two families of singularly degenerate heteroclinic cycles. Then changing slightly the parameters we numerically found a four wings butterfly shaped strange attractor.

(Some figures in this article are in colour only in the electronic version)

1. Introduction and statement of the main results

The Rabinovich system is the four-parameter family of quadratic differential equations given by

$$\dot{x} = hy - v_1x + yz, \qquad \dot{y} = hx - v_2y - xz, \qquad \dot{z} = -v_3z + xy, \qquad (1)$$

where the state variables $(x, y, z) \in \mathbb{R}^3$ and the parameters $(h, v_1, v_2, v_3) \in \mathbb{R}^4$. As usual the dots denote a derivative with respect to the time t.

System (1) was first studied in [12] throughout the analysis of a concrete realization in a magnetoactive nonisothermal plasma. From the physical point of view, it is a dynamical system of three resonantly coupled waves, parametrically excited. Numerically we get parameter

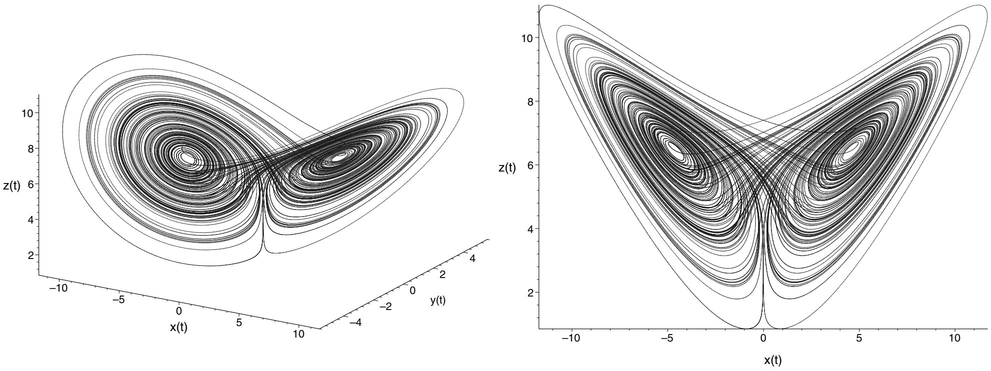

Figure 1. Rabinovich attractor (left) for the parameter values $v_1 = 4$, $v_2 = v_3 = 1$, $h = 6.75$, and its projection in the plane xy (right).

values for which a strange attractor similar to the Lorenz one is produced and it corresponds to stochastic self-oscillations of the wave amplitudes (see figure 1).

In this way, some important properties of system (1) are similar to the properties of the well-known Lorenz system (see [11])

$$\dot{x} = ry - x - yz, \qquad \dot{y} = \sigma(x - y), \qquad \dot{z} = -bz + xy. \tag{2}$$

For instance, one can easily check that: systems (1) and (2) are invariant under the change of variables $(x, y, z) \to (-x, -y, z)$, consequently if $(x(t), y(t), z(t))$ is a solution of system (1) or (2), then $(-x(t), -y(t), z(t))$, its symmetric with respect to the z-axis, is also a solution; the phase volume under the flow of both systems shrinks uniformly if $v_1 + v_2 + v_3 > 0$ and $b > r + \sigma$ because $\operatorname{div}(\dot{x}, \dot{y}, \dot{z}) = -v_1 - v_2 - v_3$ for (1), and $\operatorname{div}(\dot{x}, \dot{y}, \dot{z}) = r + \sigma - b$ for (2). Thus in these cases the attractors presented by these two systems have zero Lebesgue measure. Of course systems (1) and (2) are not topologically equivalent, since the number and local stability of their equilibria are quite different.

The integrability of system (1) has been studied using different theories and methods (see for instance [8, 13, 14]). In [3] the authors give a qualitative description of the solutions of system (1) on certain invariant algebraic surfaces. Let U be an open subset of $\mathbb{R}^3$. We remember that a *first integral* $H : U \to \mathbb{R}$ of system (1) is a function which is constant on the trajectories of the system. We say that two C^1 functions $H_1, H_2 : U \to \mathbb{R}$ are *independent* on U if the 2×3 matrix $\frac{\partial(H_1,H_2)}{\partial(x,y,z)}$ has rank 2 at all points $(x, y, z) \in U$, except perhaps on a subset of zero Lebesgue measure. System (1) is *integrable* for a choice of the parameters h, v_1, v_2, v_3 if it has two independent first integrals. If a system is integrable, then we can obtain its global phase portrait simply by performing the intersections of the level sets of its first integrals. A function $I(x, y, z, t)$ is an *invariant* of system (1) if $\mathrm{d}I/\mathrm{d}t = 0$ on the trajectories of the system, i.e. an invariant is a first integral which depends on the time. The following proposition summarizes the results on the integrability and on the existence of invariants for system (1). The proofs of these results can be found in [8, 13, 14].

Proposition 1. *System (1) has the invariants I and the first integrals H given in table 1.*

Proposition 1 will be used in the following sections for studying the global dynamics behavior of system (1) having a first integral or an invariant. We will use the Poincaré compactification for a polynomial vector field in $\mathbb{R}^3$ which is described in section 3 of [1], for $\mathbb{R}^3$ and in [7] for $\mathbb{R}^n$. We say that two polynomial vector fields X and Y on $\mathbb{R}^3$ are *topologically equivalent* if there exists a homeomorphism on the closed Poincaré ball preserving the infinity

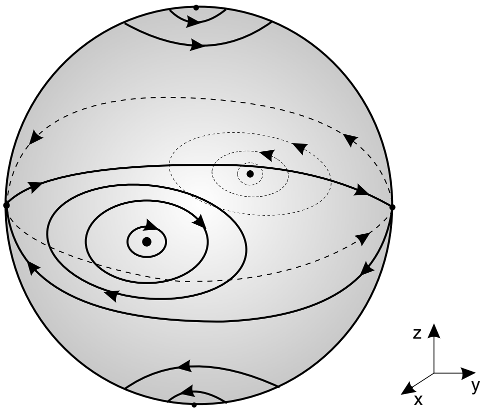

Figure 2. Phase portrait of system (1) at infinity.

Table 1. First integrals, invariants and figure with the global phase portraits of system (1). See the remarks at the end of the paper.

	v_1, v_2, v_3, h	I or H	Figure
(a)	$v_1 = v_2 = v_3 = 0$	$H_1 = x^2 + y^2 - 4hz$ $H_2 = y^2 + z^2 - 2hz$	9, 10
(b)	$v_1 = v_2 = v_3 \neq 0, h = 0$	$H = \frac{x^2 - z^2}{x^2 + y^2}$	11
(c)	$v_1 = v_2 = 0, v_3 \neq 0, h = 0$	$H = x^2 + y^2$	12
(d)	$v_2 = v_3 = 0, v_1 \neq 0$	$H = y^2 + z^2 - 2hz$	13
(e)	$v_1 = v_3 = 0, v_2 \neq 0$	$H = x^2 - z^2 - 2hz$	14, 15, 16
(f)	$v_1 = v_2 = v_3 = v > 0, h \neq 0$	$I = (x^2 - y^2 - 2z^2)\,e^{2vt}$	17
(g)	$v_1 = v_2 = v \neq 0, v_2 \neq v_3, h = 0$	$I = (x^2 + y^2)\,e^{2vt}$	18
(h)	$v_1 = v_2 = v \neq 0, v_3 = 2v, h \neq 0$	$I = (x^2 + y^2 - 4hz)\,e^{2vt}$	19
(i)	$v_2 = v_3 = v \neq 0, v_1 \neq v_2, h = 0$	$I = (y^2 + z^2)\,e^{2vt}$	20
(j)	$v_1 = v_3 = v \neq 0, v_1 \neq v_2, h = 0$	$I = (x^2 - z^2)\,e^{2vt}$	21

(i.e. the boundary of the ball) carrying orbits of the flow induced on the Poincaré ball by X into orbits of the flow induced in the Poincaré ball for Y.

The first result of this paper is the following.

Theorem 2. *For all values of the parameters h, v_1, v_2, v_3 the phase portrait of system (1) on the sphere at infinity is topologically equivalent to that shown in figure 2: there exist four centers at the positive and negative endpoints of the x- and z-axis and two hyperbolic saddles at the positive and negative endpoints of the y-axis.*

It is important to note that the dynamics at infinity does not depend on the parameter values. In this paper we study the dynamics of the Rabinovich system (1) for the systems of table 1 on the whole space $\mathbb{R}^3$ including the behavior on the sphere at infinity, i.e. on the Poincaré ball.

Theorem 3. *The global phase portrait of system (1) in the Poincaré ball for the cases (a), …,(h) of proposition 1 are topologically equivalent to those described in the figures of table 1.*

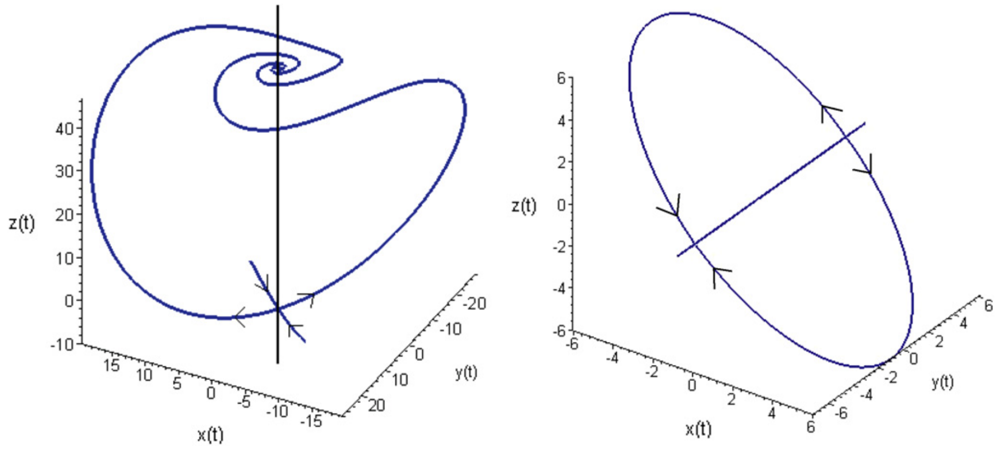

Figure 3. Singularly degenerate heteroclinic cycles: of the Lorenz system type (left) and of the Rabinovich system type (right).

In [10] the authors suggest that the existence of a *singularly degenerate heteroclinic cycle* can give the basic structure for the generation of chaotic dynamics of the Lorenz-like systems, where we can include the Rabinovich system. In that paper they get such cycles for a system equivalent to the Lorenz one, in the case of $b = 0, r \to \infty$ and bounded σ. This singular cycle consists of an invariant set formed by a line of equilibria together with a heteroclinic orbit connecting two of the equilibria (see figure 3, left). Moreover if moving the parameters near these degenerate heteroclinic cycles we have a return Poincaré map, then according with [10] a strange attractor can be created. In this paper we show that Rabinovich system also presents singularly degenerate heteroclinic cycles, which are topologically equivalent to those presented by the Lorenz system (see figure 3, right). More precisely we prove the following result.

Theorem 4. *For the parameter values $h = v_2 = 0$ and for any $v_1 = v_3 \neq 0$, system (1) has two families of singularly degenerate heteroclinic cycles. One of the families is contained in the plane $\{x = z\}$ and the other in the plane $\{x = -z\}$. Moreover each family contains an infinite set of these degenerate cycles such that when they tend to infinity they accumulate at a heteroclinic cycle on the sphere of infinity.*

As suggested in [10] we expect that some type of strange attractor should occur in a neighborhood of the families of singularly degenerate heteroclinic cycles. In fact throughout a numerical study of the solutions of the Rabinovich system we found a *four wings butterfly shaped strange attractor* for the parameter values $v_1 = 1.5, v_2 = -0.3, v_3 = 1.67$ and $h = 0.04$ (see figures 4 and 5). This attractor is different and occurs for other parameter values than that described in the literature and shown in figure 1. Four wings type attractors were recently described in [3, 4] in the study of three and four-dimensional differential systems and electronic circuits were designed which realize the phase portraits of these mathematical systems. Nice pictures of the four wings attractors have been observed as the output signal of these circuits on the oscilloscope.

The paper is organized as follows. In section 2 we summarize the results related to the dynamics and bifurcation of the finite singularities. In section 3 we prove theorem 2 by using the Poincaré compactification for a polynomial vector field in $\mathbb{R}^3$. In section 4 we prove

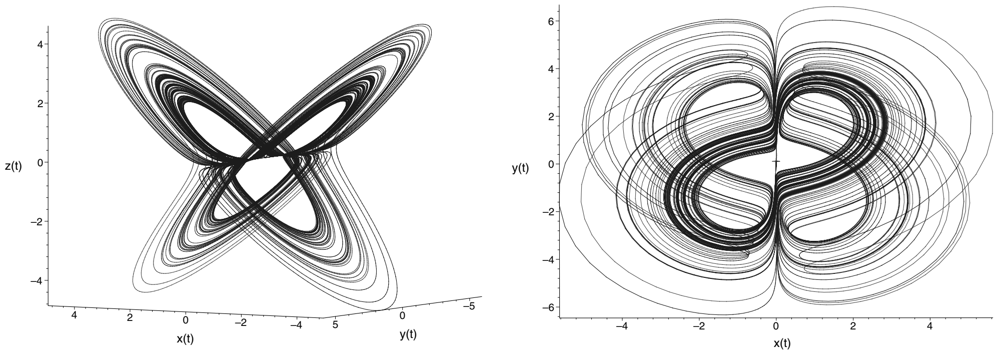

Figure 4. Four wings strange attractor for Rabinovich system (left) with parameter values $v_1 = 1.5$, $v_2 = -0.3$, $v_3 = 1.67$, $h = 0.04$ and its projection on the plane xy (right).

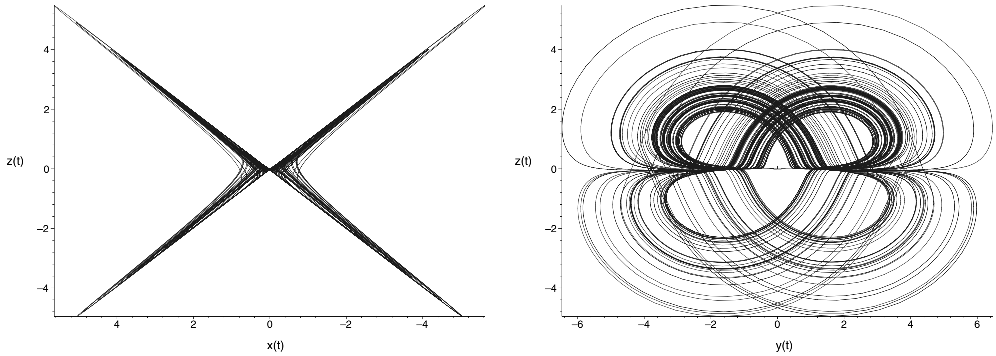

Figure 5. Projections of figure 4, left, on the planes xz (on the left) and yz (on the right).

theorem 3. Finally, in section 5 we prove theorem 4 and numerically we compute some strange attractors in a neighborhood of the families of singularly degenerate heteroclinic cycles.

2. The dynamics of finite singularities

In this section we present a result on the α- and the ω-limit sets of a bounded trajectory, and summarize the results related to the dynamics and bifurcation of the finite singularities of Rabinovich system.

Lemma 5. *If $\varphi(t) = (x(t), y(t), z(t))$, $t \in \mathbb{R}$, is a bounded trajectory of Rabinovich system (1) satisfying one of the conditions (g),. . .,(j) of table 1 with invariant given by $I = f \cdot \mathrm{e}^{2vt}$, then its α-limit set $\alpha(\varphi)$ and its ω-limit set $\omega(\varphi)$ are contained in the set $\mathcal{S} = \{f = 0\}$.*

Proof. Suppose that $v > 0$. Let $q_1 \in \omega(\varphi)$. Thus there exists $t_n \to \infty$ such that $\varphi(t_n) \to q_1$. Thus $f(\varphi(t_n)) = \frac{c}{\mathrm{e}^{vt_n}} \to 0$ and $f(\varphi(t_n)) \to f(q_1)$. It follows that $f(q_1) = 0$ and then $q_1 \in \mathcal{S}$. Suppose now that $q_2 \in \alpha(\varphi)$. Thus there exists $t_n \to -\infty$ such that $\varphi(t_n) \to q_2$. It follows that $f(\varphi(t_n)) \cdot \mathrm{e}^{vt_n} \to f(q_2) \cdot 0 = c$. Thus $c = 0$ and then $f(\varphi(t_n)) \cdot \mathrm{e}^{vt_n} = 0$. It implies that $f(\varphi(t_n)) = 0$ and thus $f(q_2) = 0$. Then $q_2 \in \mathcal{S}$. □

Proposition 6. *Let $X_{h,v} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field corresponding to system (1), with $v_1 = v_2 = v_3 = v$, that is the vector field given by $X_{h,v}(x, y, z) = (hy - vx + yz, hx - vy - xz, -vz + xy)$.*

(a) The singularities of $X_{0,0}$ are given by the three coordinate axes. If $h \neq 0$ then $X_{h,0}$ is linearly equivalent to $X_{1,0}$. The singularities of $X_{1,0}$ are given by the following straight lines: $(0, 0, z)$ with $z \in \mathbb{R}$; $(x, 0, 1)$ with $x \in \mathbb{R}$; and $(0, y, -1)$ with $y \in \mathbb{R}$. The solutions of $X_{1,0}$ and $X_{0,0}$ belong to the intersection of the level sets $H_1 = k_1$ and $H_2 = k_2$ of the first integrals given in table 1(a).

(b) If $v \neq 0$ then $X_{h,v}$ is linearly equivalent to $X_{h/v,1}$. If $H = \frac{h}{v} \neq 0$ and $|H| \leqslant 1$ then the only singular point of $X_{H,1}$ is the origin $S_0 = (0, 0, 0)$. If $|H| < 1$ it is an attracting node or focus. For $|H| = 1$ a pitchfork bifurcation occurs, that is S_0 becomes non-hyperbolic since one of its eigenvalues has zero real part, and for $|H| > 1$ it becomes a saddle and two new symmetric singular points $S_\pm$ appear, where

$$S_\pm = \left(\pm \frac{HR}{H^2 - \sqrt{-H^2 + H^4}}, \pm R, \frac{HR^2}{H^2 - \sqrt{-H^2 + H^4}} \right),$$

with $R = \sqrt{-H^2 + 1 + \sqrt{-H^2 + H^4}}$. The singular points $S_\pm$ are attracting nodes or foci. If $\varphi(t) = (x(t), y(t), z(t))$, $t \in \mathbb{R}$, is a bounded trajectory of $X_{H,1}$ then its α-limit set $\alpha(\varphi)$ and its ω-limit set $\omega(\varphi)$ are contained in the cone $\mathcal{C} = \{x^2 - y^2 - 2z^2 = 0\}$. In particular $S_0, S_+, S_- \in \mathcal{C}$. Moreover, for $H = 0$ we have that S_0 is the unique singular point of $X_{0,1}$.

Proof. The proof of statement (a) is easy. Now we prove statement (b). Suppose that $h \neq 0$ and that $v = 0$. Thus the linear change of coordinates $x \to hX$, $y \to hY$, $z \to hZ$ and the time rescaling $t = h\tau$ give the linear equivalence. Similar argument can be used if $v = 0$. The singularities of $X_{H,1}$ are given by $\left(\frac{HR}{1-R^2}, R, \frac{HR^2}{1-R^2}\right)$, with R satisfying the equation $R^4 - 2(1-H^2)R^2 + 1 - H^2 = 0$. Solving this equation we get $R = \sqrt{-H^2 + 1 + \sqrt{-H^2 + H^4}}$. Thus there exists $R \neq 0$ only if $|H| > 1$ and in this case $-H^2 + 1 - \sqrt{-H^2 + H^4} < 0$. To classify the singular points we compute the eigenvalues of the linear part $JX_{H,1}(S_0)$ and find $\lambda_1 = -H - 1$, $\lambda_2 = -1$, and $\lambda_3 = H - 1$. The eigenvalues of the linear parts $JX_{H,1}(S_+)$ and $JX_{H,1}(S_-)$ are $-2, -\frac{1}{2} + \frac{\sqrt{9-8H^2}}{2}, -\frac{1}{2} - \frac{\sqrt{9-8H^2}}{2}$. According to table 1(g) there exists $c \in \mathbb{R}$ such that

$$I(\varphi(t), t) = (x^2(t) - y^2(t) - 2z^2(t)) \cdot e^{2t} = g(\varphi(t)) \cdot e^{2t} = c.$$

The assertions about the ω-limit and about the α-limit follow from lemma 5. □

Following the computations of [3] one can see that the Rabinovich system $X_{H,1}$, with $H \neq 0$, restricted to the surface $\mathcal{C}$ with $x \geqslant 0$ is

$$\dot{r} = r(-1 + H\cos\theta + r\sin\theta\cos\theta), \qquad \dot{\theta} = r(1 + \cos^2\theta) - H\sin\theta, \tag{3}$$

where $x = \sqrt{2}r$, $y = \sqrt{2}r\cos(\theta)$, and $z = r\sin(\theta)$. System (3) has no finite periodic orbits and its infinity is an unstable limit cycle. Moreover the flow on the negative half-cone ($x \leqslant 0$) is obtained by using the symmetry $(x, y, z) \to (-x, -y, z)$.

Proposition 7. *Let $X_{h,v_1,v_2,v_3} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field corresponding to system (1). If $v \neq 0$ then $X_{h,v,v,2v}$ is linearly equivalent to $X_{h/v,1,1,2}$. Consider $H = \frac{h}{v}$.*

(a) If $H \neq 0$ and $|H| \leqslant 1$, then the only singular point of $X_{H,1,1,2}$ is $L_0 = (0, 0, 0)$. Moreover if $|H| < 1$ it is an attracting node or focus, and if $|H| = 1$ it is not a hyperbolic singular point.

(b) If $H \neq 0$ and $|H| > 1$, then the singular points of $X_{H,1,1,2}$ are L_0, L_+ and L_- where $L_\pm = \left(\pm \frac{T(2H^2+2\sqrt{H^4-H^2})}{2H}, \pm T, \frac{\sqrt{H^4-H^2}}{H}\right)$, and $T = \sqrt{-2H^2+2+2\sqrt{H^4-H^2}}$. The singular point L_0 is a saddle with stability index 2 (i.e, two eigenvalues with negative real part and one eigenvalue with positive real part) and the singular points L_+ and L_- are attracting nodes or foci. If $\varphi(t) = (x(t), y(t), z(t))$, $t \in \mathbb{R}$ is a bounded trajectory then its α-limit set and its ω-limit set are contained in the surface $\mathcal{P} = \{x^2 + y^2 - 4Hz = 0\}$. In particular $L_0, L_+, L_- \in \mathcal{P}$.

(c) L_0 is the only singular point of $X_{0,1,1,2}$.

Proof. The eigenvalues of the linear part $JX_{H,1,1,2}(L_0)$ are $\lambda_1 = -2, \lambda_2 = H - 1, \lambda_3 = -H - 1$. If $|H| > 1$ then the eigenvalues of the linear parts $JX_{H,1,1,2}(L_+)$ and $JX_{H,1,1,2}(L_-)$ are $-2, -1 + \sqrt{5-4H^2}, -1 - \sqrt{5-4H^2}$. □

Following the computations of [3], the Rabinovich system $X_{H,1,1,2}$ restricted to the surface $\mathcal{P}$ is

$$\dot{x} = -x + Hy + \frac{1}{4H}y(x^2+y^2), \qquad \dot{y} = Hx - y - \frac{1}{4H}x(x^2+y^2). \tag{4}$$

Since the divergence of (4) is -2, one can see that it has no limit cycles nor loops containing finitely many finite singularities.

Proposition 8. *Let $X_{h,v_1,v_2,1} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field corresponding to system (1) for the case $v_3 = 1$. If $h < \sqrt{v_1 v_2}$ then $C_0 = (0, 0, 0)$ is the only singular point. If $h \geqslant \sqrt{v_1 v_2}$ then it has three singular points: C_0, C_+ and C_- where $C_\pm = \left(\pm \frac{hK}{v_1 - R^2}, \pm K, \frac{hK^2}{v_1 - K^2}\right)$, and $K = \sqrt{\frac{-h^2+v_1v_2+\sqrt{h^4-h^2v_1v_2}}{v_2}}$. The point C_0 is a saddle with stability index 2.*

The proof of proposition 8 is easy and it is omitted.

A numerical investigation of system (1) is done at [12]. For $v_1 = 4$, $v_2 = 1$, $v_3 = 1$ and $h = 6.75$ the singular points are $C_0 = (0, 0, 0)$, $C_\pm = (\pm 4.611\ldots, \pm 1.397\ldots, 6.446\ldots)$. The eigenvalues of the linear part at C_0 are $-9.414\ldots, -1, 4.414\ldots$, and at $C_\pm$ are $-6.316\ldots \pm i(5.127\ldots)$. The trajectory of a point makes several revolutions around the singular point C_+ then goes over to C_- and rotates around it, returns back to C_+, etc. One can see that for $v_1 = 4$ and $v_2 = v_3 = 1$ there are values of h such that limit cycles and strange attractors appear (see figure 1). This analysis can be obtained with all details in [12].

3. The behavior on the sphere at infinity

In this section we do an analysis of the flow of system (1) at infinity. Fixing the notation in accordance with the results stated in [1] we write the polynomial differential system (1) as $\dot{x} = P^1$, $\dot{y} = P^2$ and $\dot{z} = P^3$ with

$$P^1 = hy - v_1x + yz, \qquad P^2 = hx - v_2y - xz, \qquad P^3 = -v_3z + xy.$$

In the next four subsections we study the Poincaré compactification of system (1) in the local charts U_i and V_i, $i = 1, 2, 3$.

3.1. In the local charts U_1 and V_1

The Poincaré compactification $p(X)$ of system (1) in the local chart U_1 is given by

$$\begin{aligned}
\dot{z}_1 &= -z_2 + hz_3 - hz_1^2z_3 - z_2z_1^2 + v_1z_1z_3 - v_2z_1z_3,\\
\dot{z}_2 &= z_1 - hz_1z_2z_3 - z_2^2z_1 + v_1z_2z_3 - v_3z_2z_3,\\
\dot{z}_3 &= z_3(-hz_1z_3 + v_1z_3 - z_1z_2).
\end{aligned} \tag{5}$$

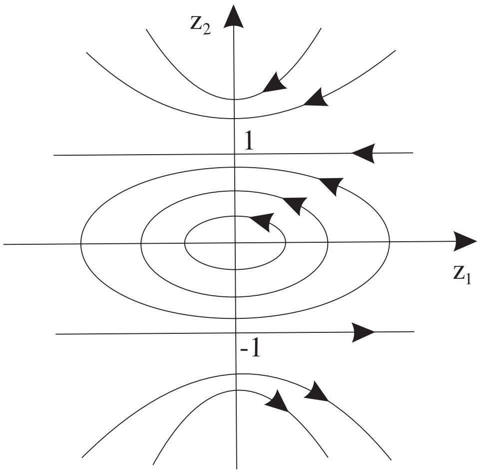

Figure 6. Phase portrait of system (6).

For $z_3 = 0$ (which corresponds to the points of the sphere $\mathbb{S}$ at infinity) the unique singular point of (5) is $(0, 0, 0)$ and the eigenvalues of the linear part of the system at this point are i, $-$i and 0, the zero eigenvalue has eigenvector $(0, h, 1)$.

In general the dynamics near a non-hyperbolic singular point of this type can be rather complex, see for instance [9]. Fortunately as a property of the compactification procedure, the plane z_1z_2 is invariant under the flow of system (5), which makes the analysis on the infinite sphere simpler. Taking $z_3 = 0$ the equations of system (5) reduce to

$$\dot{z}_1 = -z_2 - z_1^2 z_2, \qquad \dot{z}_2 = z_1 - z_1 z_2^2, \tag{6}$$

which have $(0, 0)$ as their unique singular point and the eigenvalues of the linear part of the system at this point are given by $\pm$i. Hence the point is a focus or a center. Since system (6) has the first integral $H = \ln\left(1 + z_1^2\right)^2 - \ln\left(1 - z_2^2\right)^2$, the origin is a center. Using this first integral and observing that system (6) has $z_2 = \pm 1$ as invariant lines, it follows that the phase portrait on the local chart U_1 on the infinite sphere is as shown in figure 6. The flow on the local chart V_1 is the same than the flow on the local chart U_1 reversing the time, because the compactified vector field $p(X)$ in V_1 coincides with the vector field $p(X)$ in U_1 multiplied by -1 (see [1]).

3.2. In the local charts U_2 and V_2

The expression of the Poincaré compactification $p(X)$ of system (1) in the local chart U_2 is given by

$$\begin{aligned} \dot{z}_1 &= z_2 + hz_3 + z_1^2 z_2 - hz_1^2 z_3 + v_2 z_1 z_3 - v_1 z_1 z_3, \\ \dot{z}_2 &= z_1 + z_1 z_2^2 - hz_1 z_2 z_3 + v_2 z_2 z_3 - v_3 z_2 z_3, \\ \dot{z}_3 &= v_2 z_3^2 - hz_1 z_3^2 + z_1 z_2 z_3. \end{aligned} \tag{7}$$

If $z_3 = 0$ the unique singular point of system (7) is $p_1 = (0, 0, 0)$. We want to study the local flow of this system at p_1. The eigenvalues of the linear part of (7) at p_1 are -1, 1 and 0, with eigenvectors $(1, 1, 0)$, $(-1, 1, 0)$ and $(0, -h, 1)$, respectively. Hence system (7) has a two-dimensional saddle at p_1 when we restrict the flow to the infinity (i.e. to $z_3 = 0$, which is invariant) and a one-dimensional center manifold to p_1 contained in the interior of the ball diffeomorphic to $\mathbb{R}^3$. We are interested in the flow of system (7) for $z_3 > 0$, which enables us to study the flow of system (1) in the positive z-axis near the point (0, 0, 0) at infinity. To understand the dynamics of the system we shall study the flow on this center manifold. The following result holds.

Proposition 9. *The singular point $p_1 = (0, 0, 0)$ of system (7) is asymptotically stable (resp. unstable) along a one-dimensional center manifold to p_1 if $v_2 < 0$ (resp. $v_2 > 0$).*

Proof. Under the above considerations, from the center manifold theorem (see [2] or [6]) it follows that system (7) has one-dimensional center manifold at the singular point $p_1 = (0, 0, 0)$, which is the graph of a function $h : \mathbb{R} \to \mathbb{R}^2$ given by $(z_1, z_2) = F(z_3) = (F_1(z_3), F_2(z_3))$ satisfying the conditions $F(0) = (0, 0)$, $DF(0) = (0, -h)$, and

$$\dot{z}_1 - DF_1(z_3)\dot{z}_3 = 0, \qquad \dot{z}_2 - DF_2(z_3)\dot{z}_3 = 0. \tag{8}$$

Moreover the flow on this center manifold is governed by the one-dimensional equation

$$\dot{z}_3 = v_2 z_3^2 + F_1(z_3)F_2(z_3)z_3 - hF_1(z_3)z_3^2. \tag{9}$$

To understand the above flow on this manifold we expand F in Taylor series around $z_3 = 0$. Using the conditions, we can take

$$F_1(z_3) = \sum_{i=2}^{\infty} a_i z_3^i \qquad \text{and} \qquad F_2(z_3) = -hz_3 + \sum_{i=2}^{\infty} b_i z_3^i. \tag{10}$$

Now from (8) and considering the expressions for $\dot{z}_1$ and $\dot{z}_2$ given by system (7) we have

$$F_2 + hz_3 + F_1^2F_2 - hF_1^2z_3 + v_2F_1z_3 - v_1F_1z_3 - DF_1\left(v_2z_3^2 + F_1F_2z_3 - hF_1z_3^2\right) = 0, \tag{11}$$

$$F_1 + F_1F_2^2 - hF_1F_2z_3 + v_2F_2z_3 - v_3F_2z_3 - DF_2\left(v_2z_3^2 + F_1F_2z_3 - hF_1z_3^2\right) = 0, \tag{12}$$

where $F_1 = F_1(z_3)$, $F_2 = F_2(z_3)$ are provided by (10) and $DF_i = DF_i(z_3) = F_i'(z_3), i = 1, 2$. Equating the coefficients of the powers of z_3 in (11) and in (12) one obtains

$$a_2 = h(v_2 - v_3), \quad a_3 = 0, \quad a_4 = h(v_2 - v_3)[-2h + v_3(v_1 - v_2) - v_2(v_1 - v_2)] + v_2h$$

and

$$b_2 = 0, \qquad b_3 = h(v_2 - v_3)(v_1 - v_2), \qquad b_4 = 0.$$

Hence an approximation of the local center manifold at p_1 is the graph of the function $(z_1, z_2) = F(z_3) = (F_1(z_3), F_2(z_3))$, where

$$F_1(z_3) = h(v_2 - v_3)z_3^2 + \{h(v_2 - v_3)[-2h + v_3(v_1 - v_2) - v_2(v_1 - v_2)] + v_2h\}z_3^4$$

and $F_2(z_3) = h(v_2 - v_1)(v_1 - v_2)z_3^3$. Now it follows from (9) that the flow on this center manifold is determined by the equation $\dot{z}_3 = v_2z_3^2 + O\left(z_3^4\right)$ which implies that, for $z_3 > 0$, p_1 is locally asymptotically stable along its center manifold if $v_2 < 0$, and it is unstable if $v_2 > 0$. □

From proposition 9 one can conclude that there is a trajectory of system (1) which escapes to infinity as $t \to \pm\infty$ (depending on the sign of v_2). In fact, since the infinity of this system in the local chart U_2 is invariant, the unique way in order that a solution reaches the infinity is tending to the singular point p_1 (which is a saddle on the sphere at infinity) and this is possible only over the center manifold. The flow on the local chart V_2 is the same as the flow on the local chart U_2 reversing the time, because the compactified vector field $p(X)$ in V_2 coincides with the vector field $p(X)$ in U_2 multiplied by -1.

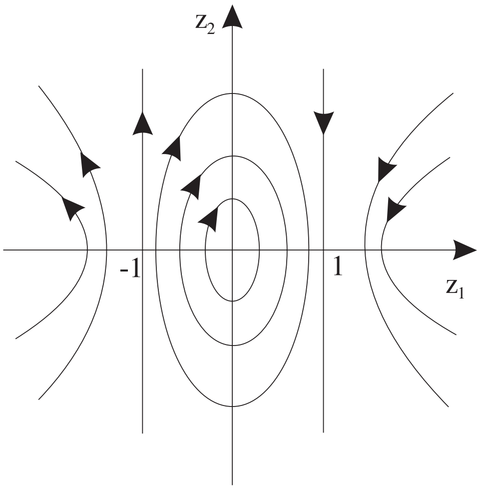

Figure 7. Phase portrait of system (14).

Remark 10. If $h = 0$ then the above center manifold coincides with the z_3-axis. Indeed, in this case if we consider $z_1 = z_2 = 0$ system (7) reduces to $\dot{z}_1 = 0$, $\dot{z}_2 = 0$, $\dot{z}_3 = v_2 z_3^2$. Hence considering $z_3 > 0$ the origin in the local chart U_2 is stable (resp. unstable) if $v_2 < 0$ (resp. $v_2 > 0$). In the local chart V_2 we have to change the sign of system (7), so in this chart the dynamics on the z_3-axis is governed by the equation $\dot{z}_3 = -v_2 z_3^2$. Considering $z_3 < 0$ (see [1]) we have that the origin is stable (resp. unstable) if $v_2 < 0$ (resp. $v_2 > 0$).

3.3. In the local charts U_3 and V_3

The expression of the Poincaré compactification $p(X)$ in the local chart U_3 is

$$\begin{aligned} \dot{z}_1 &= z_2 - z_1^2 z_2 + h z_2 z_3 + v_3 z_1 z_3 - v_1 z_1 z_3, \\ \dot{z}_2 &= -z_1 + h z_1 z_3 - z_1 z_2^2 + v_3 z_2 z_3 - v_2 z_2 z_3, \\ \dot{z}_3 &= v_3 z_3^2 - z_1 z_2 z_3. \end{aligned} \tag{13}$$

Again the eigenvalues of the linear part of (13) at (0, 0, 0) which is the unique singular point on $z_3 = 0$, are i, −i and 0, and the zero eigenvalue has eigenvector (0, 0, 1). Taking $z_3 = 0$ the equations of system (13) reduce to

$$\dot{z}_1 = z_2 - z_1^2 z_2, \qquad \dot{z}_2 = -z_1 - z_1 z_2^2, \tag{14}$$

which has (0, 0) as its unique singular point. System (14) has the first integral $H = \left(z_1^2 - 1\right)/\left(z_2^2 + 1\right)$. Hence the origin of the chart U_1 restricted to the infinite sphere is a center. Using this first integral and observing that system (6) has $z_1 = \pm 1$ as invariant lines, it follows that the phase portrait on the local chart U_1 over the infinite sphere is as shown in figure 7.

Again we observe that the flow on the local chart V_3 is the same as the flow on the local chart U_3 reversing the time, because the compactified vector field $p(X)$ in V_3 coincides with the vector field $p(X)$ in U_3 multiplied by −1.

Proof of theorem 2. Considering the analysis made in the previous subsections we have a global picture of the system (1) on the sphere at infinity: the system has four centers, localized

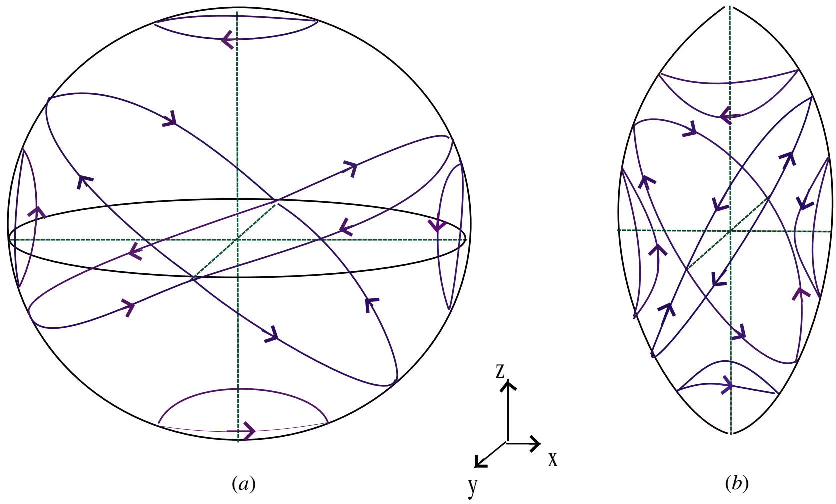

Figure 8. Phase portrait of the Rabinovich system with $v_1 = v_2 = v_3 = h = 0$ on $\mathbb{S}^2$ in (*a*), and on a cylinder $x^2 + y^2 = c_1$ in (*b*).

at the endpoints of the x- and z-axis, and two saddles, localized at the endpoints of the y-axis (see figure 2). Moreover, the orbits of the system may come from and go to infinity along a one-dimensional center manifold of these saddle points, depending on the sign of the parameter v_2. The dynamics near the center at infinity is more complex, due to the periodic orbits. □

4. Global phase portraits of the Rabinovich system

In this section we prove theorem 3. We consider the invariants and the first integrals given in table 1 and how these surfaces end in the Poincaré sphere at infinity.

4.1. Case $v_1 = v_2 = v_3 = 0$ (table 1(a))

For these parameter values the Rabinovich system is completely integrable.

If $h = 0$ the axes x, y and z are formed by singular points. According to proposition 1 the trajectories are contained in the intersection of the cylinders $x^2 + y^2 = c_1$ and $y^2 + z^2 = c_2$, where $c_1, c_2 \geqslant 0$. For $c_1 \neq c_2$ the intersection is formed by two periodic orbits and if $c_1 = c_2$ the intersection is formed by four heteroclinic cycles which share a common segment of singular points on the y-axis. Near the infinity the periodic orbits approach the orbits of the centers at infinity and the heteroclinic cycles approach the saddle separatrices at infinity. See figure 8.

If $h \neq 0$ then according to proposition 6(a) it is enough to consider $h = 1$. For these parameter values the Rabinovich system is completely integrable. The singularities are given by the z-axis and the straight lines $(x, 0, 1)$ and $(0, y, -1)$. The endpoints of $(x, 0, 1)$ on the Poincaré sphere are $(\pm 1, 0, 0)$, and of $(0, y, -1)$ are $(0, \pm 1, 0)$. According to proposition 1 the trajectories are in the intersection of the paraboloids $x^2 + y^2 - 4z = c_1$ with the cylinders $y^2 + (z - 1)^2 = c_2$ where $c_1, c_2 \in \mathbb{R}$. The intersection produces families of periodic orbits, and for $c_1 = c_2$ the intersection occurs on the planes $x = \pm(z + 1)$. See figure 9.

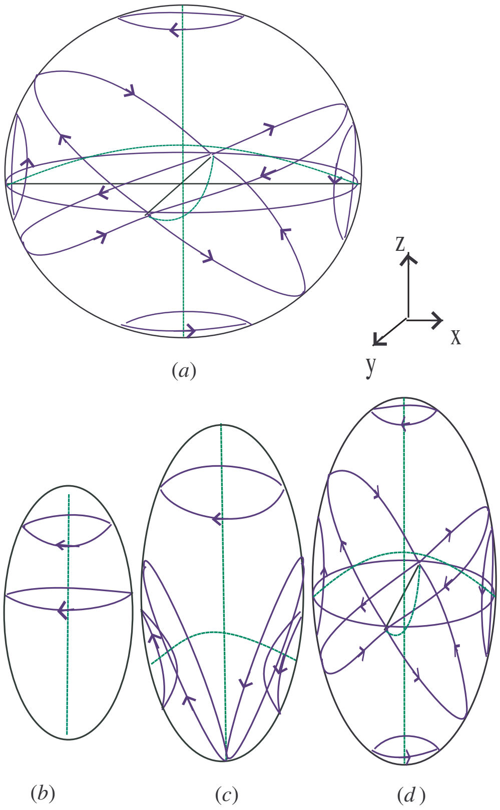

Figure 9. Phase portrait of the Rabinovich system with $v_1 = v_2 = v_3 = 0, h \neq 0$ on $\mathbb{S}^2$ in (*a*), and on the paraboloids $x^2 + y^2 - 4z = c_1$ with $c_1 \in (-\infty, -4]$ in (*b*), with $c_1 \in (-4, 4]$ in (*c*), and with $c_1 \in (4, \infty)$ in (*d*).

4.2. Case $v_1 = v_2 = v_3 = v \neq 0, h = 0$ *(table 1(b))*

According to proposition 6(b) it is enough to consider $v = 1$. In this case the trajectories are in the cone $(1 - k)x^2 - ky^2 - z^2 = 0$ with $k \in \mathbb{R}\backslash\{0\}$ and in the planes $x = \pm z$ when $k = 0$. The $(0, 0, 0)$ is the only singular point which is an attracting node. Moreover the divergence is constant and equal to -3, and consequently the system has no finite periodic orbits. The three axes are invariant. See figure 10.

4.3. Case $v_1 = v_2 = 0, v_3 \neq 0, h = 0$ *(table 1(c))*

In this case $H = x^2 + y^2$ is a first integral. The z-axis is formed by singular points and a complete analysis of the dynamics of the Rabinovich system restricted to the cylinder $x^2 + y^2 = r^2$ with $r > 0$ can be found in [3]. For $v_3 < 0$, in each cylinder there are two stable nodes or foci and two saddles. Moreover the system has no limit cycles and loops containing finitely many singularities in the invariant cylinders. Observe that the endpoints of the cylinders are the points $(0, 0, \pm 1)$ of the Poincaré sphere. For $v_3 > 0$ the phase portrait on the cylinder is the same reversing the time. See figure 11.

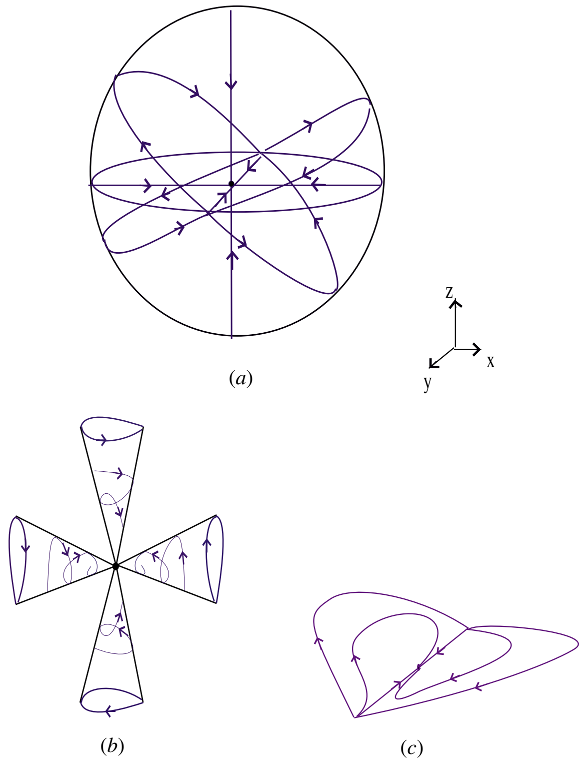

Figure 10. Phase portrait of the Rabinovich system with $v_1 = v_2 = v_3 \neq 0$, $h = 0$ on $\mathbb{S}^2$ in (*a*), and on the cone $(1-k)x^2 - ky^2 - z^2 = 0$ $(k \neq 0)$ in (*b*), and on the planes $x = z$ $(k = 0)$ with $z > 0$ in (*c*).

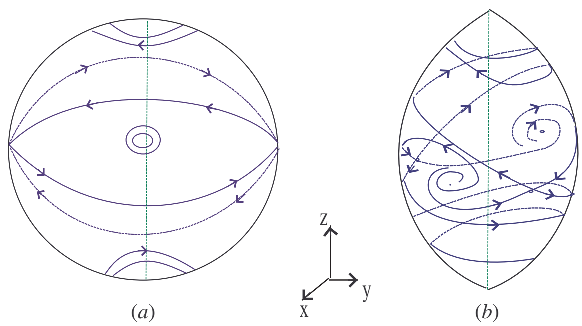

Figure 11. Phase portrait of the Rabinovich system with $v_1 = v_2 = 0$, $v_3 \neq 0$, $h = 0$ on $\mathbb{S}^2$ in (*a*), and on a cylinder $x^2 + y^2 = r^2$ in (*b*).

4.4. Case $v_2 = v_3 = 0$, $v_1 \neq 0$ (table 1(d))

In this case $H = y^2 + z^2 - 2hz$ is a first integral. A complete analysis of the dynamics of the Rabinovich system restricted to the level $y^2 + (z-h)^2 = r^2$ with $r > 0$ can be found in [3]. The straight line $(x, 0, h)$ is formed by singular points. Observe that the endpoints of the

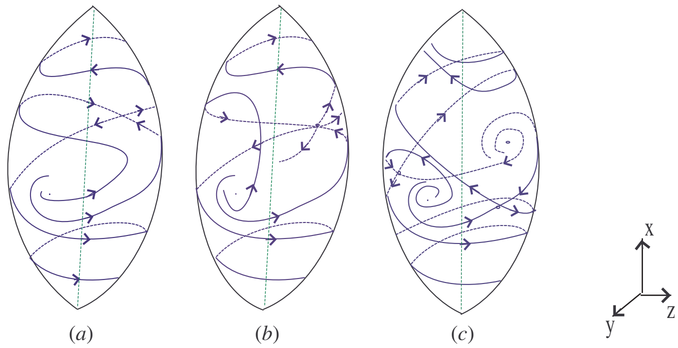

Figure 12. Global phase portrait of Rabinovich system with $v_2 = v_3 = 0$, $v_1 \neq 0$ on the cylinders $y^2 + (z - h)^2 = r^2$ with $r < 2h$ in (*a*), with $r = 2h$ in (*b*) and with $r > 2h$ in (*c*).

cylinders are the points $(\pm 1, 0, 0)$ of the Poincaré sphere. If $v_1 > 0$ and $h \geqslant 0$ the system has two singularities when $r < 2h$ (an unstable node or focus and a saddle) and four singularities when $r > 2h$ (two nodes or foci and two saddles). For $r = 2h$ three singularities coincide. Moreover the system has no periodic orbits and loops containing finitely many singularities on the invariant cylinders. A complete phase portrait is the union of the figures 12(*a*)–(*c*), with the sphere. For $v_1 < 0$ the phase portrait on the cylinders is the same reversing the time.

4.5. Case $v_1 = v_3 = 0, v_2 \neq 0$ *(table 1(e))*

In this case $H = x^2 - (z + h)^2$ is a first integral. Assume that $v_2 > 0$. A complete analysis of the dynamics of the Rabinovich system restricted to the hyperbolic cylinders $x^2 - (z + h)^2 = \pm r^2$, if $r \neq 0$, and on the planes $x^2 - (z + h)^2 = 0$ can be found in [3]. For $r = 0$ the level is formed by two planes with endpoints being the saddle separatrices on the infinity sphere. The dynamics on the planes is the following. The system has two singular points, an attracting node or focus, and the $(0, 0, 0)$ which is a non-hyperbolic singular point. The straight line $(0, y, -h)$ (intersection of the planes $x = z + h$ and $x = -(z + h)$) is invariant with its points having $(0, 0, 0)$ like ω-limit. See figure 13. For $r \neq 0$ the level is a pair of hyperbolic cylinders. On each hyperbolic cylinder $x^2 - (z + h)^2 = -r^2, r \neq 0$ there exists three singular points (a saddle and two attracting nodes or foci) if $r < 2h$, and an attracting node or focus if $r \geqslant 2h$. On each hyperbolic cylinder $x^2 - (z + h)^2 = r^2, r \neq 0$ there exists only one singular point which is an attracting node or focus. For $v_2 < 0$ the phase portrait on the cylinder is the same reversing the time. A complete phase portrait can be obtained with the union of the figures 13–15 and the infinity sphere.

4.6. Case $v_1 = v_2 = v_3 = v > 0, h \neq 0$ *(table 1(f))*

According to proposition 6(b) it is enough to consider $v = 1$. In this case the cone $x^2 = y^2 + 2z^2$ is an invariant algebraic surface. The z-axis is invariant. Since the divergence is constant -3 it follows that the system has no finite periodic orbits. If $0 \neq |h| \leqslant 1$ then $(0, 0, 0)$ is the only singular point and it is an attracting node or focus if $|h| < 1$. If $|h| > 1$ then the system has three singular points, a saddle $(0, 0, 0)$, and two attracting nodes or foci $S_\pm$. The singularity S_+ is on the part of the cone with $x > 0$ and S_- is on the part of the cone with $x < 0$. Two separatrices (one stable and one unstable) of the saddle are in the part of the cone with $x > 0$ and the other two are in the part with $x < 0$. See figure 16. According

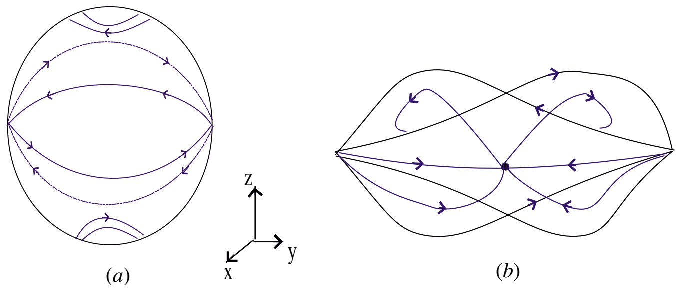

Figure 13. Phase portrait of the Rabinovich system for $v_1 = v_3 = 0, v_2 \neq 0, h \geqslant 0$ on $\mathbb{S}^2$ in (*a*), and on the planes $x^2 - (z+h)^2 = 0$ in (*b*).

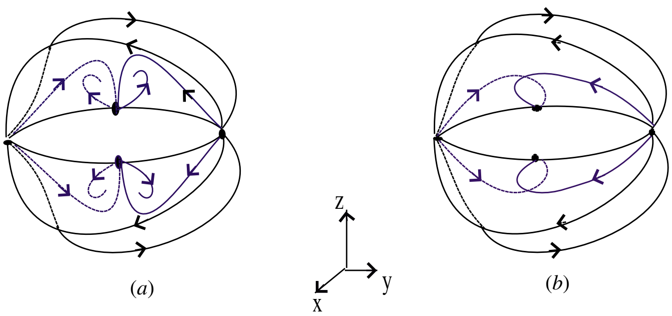

Figure 14. Phase portrait of the Rabinovich system for $v_1 = v_3 = 0, v_2 \neq 0, h \geqslant 0$ on the hyperbolic cylinder $x^2 - (z-h)^2 = -r^2$ with $r < 2h$ in (*a*), and with $r \geqslant 2h$ in (*b*).

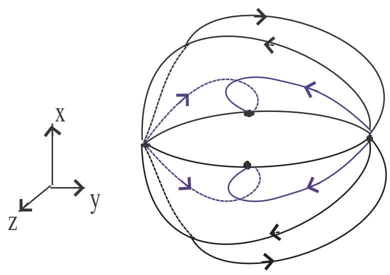

Figure 15. Phase portrait of the Rabinovich system for $v_1 = v_3 = 0, v_2 \neq 0, h \geqslant 0$ on the hyperbolic cylinder $x^2 - (z-h)^2 = r^2$.

to lemma 5 for any regular trajectory γ not starting on the cone we have that $\alpha(\gamma) \subset \mathcal{C}$ and $\omega(\gamma) \subset \mathbb{S}^2 \cup \mathcal{C}$.

4.7. *Case* $v_1 = v_2 = v \neq 0, v_2 \neq v_3, h = 0$ *(table 1(g))*

For these parameters the only information that we have is that $I = (x^2+y^2)\,\mathrm{e}^{2vt}$ is an invariant. The Rabinovich system restricted to $(0, 0, z)$ (i.e. $f = x^2 + y^2 = 0$) is the trivial linear system $\dot{z} = -v_3 z$. If $v < 0$ and $v_3 > 0$, then the ω–limit of any orbit γ is $\omega(\gamma) = \{(0, 0, 0)\}$, and the

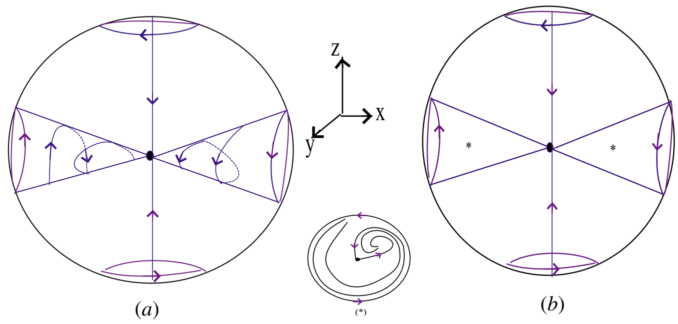

Figure 16. Phase portrait of the Rabinovich system for $v_1 = v_2 = v_3 = 1, h \geqslant 0$ on the cone $x^2 = y^2 + 2z^2$ with $0 \neq |h| \leqslant 1$ in (*a*), and with $|h| > 1$ in (*b*).

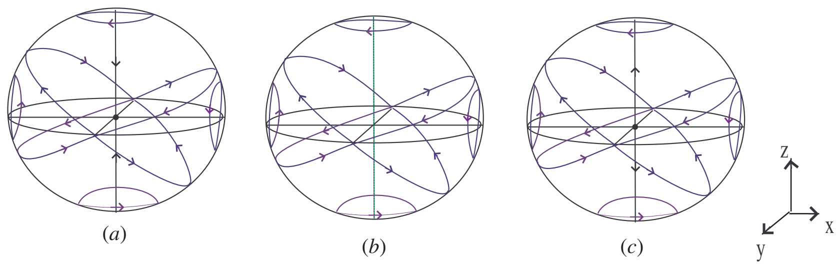

Figure 17. Phase portrait of the Rabinovich system for $v_1 = v_2 = v \neq 0$, $v_2 \neq v_3$, $h = 0$ on the sphere with $v_3 > 0$ in (*a*), with $v_3 = 0$ in (*b*), and with $v_3 < 0$ in (*c*).

α–limit is contained on $\mathbb{S}^2$. If $v < 0$ and $v_3 < 0$, then the ω-limit of any orbit γ is $\omega(\gamma) \subset \mathbb{S}^2$ and the α-limit is contained on $\mathbb{S}^2$, too. If $v < 0$ and $v_3 = 0$, then the ω-limit of any orbit γ is $\omega(\gamma) \subset z$-axis. See figure 17.

4.8. *Case* $v_1 = v_2 = v \neq 0$, $v_3 = 2v$, $h \neq 0$ *(table 1(h))*

According to proposition 7 it is enough to consider $v = 1$ and $h > 0$. In this case the paraboloid $x^2 + y^2 - 4hz = 0$ is an invariant algebraic surface. A complete analysis of the dynamics of the Rabinovich system restricted to $\mathcal{P}$ can be found in [3]. It has no finite periodic orbits. If $0 < h \leqslant 1$ then $(0, 0, 0)$ is the only singular point and it is an attracting node or focus (a weak focus if $h = 1$). If $h > 1$ then it has three singular points a saddle $(0, 0, 0)$, and two attracting node or foci $L_\pm$. The endpoints of the paraboloid is $(0, 0, 1)$. According to lemma 5 for any trajectory γ not contained in the paraboloid we have that $\alpha(\gamma) \subset \mathcal{P}$ and $\omega(\gamma) \subset \mathbb{S}^2$. See figure 18.

4.9. *Case* $v_2 = v_3 = v \neq 0$, $v_1 \neq v_2$, $h = 0$ *(table 1(i))*

For these parameters the only information that we have is that $I = (y^2 + z^2)\,e^{2vt}$ is an invariant. The Rabinovich system restricted to $(x, 0, 0)$ is the trivial linear system $\dot{x} = -v_1 x$. If $v < 0$ and $v_1 > 0$, then the ω-limit of any orbit γ is $\omega(\gamma) = \{(0, 0, 0)\}$, and the α-limit is contained on $\mathbb{S}^2$. If $v < 0$ and $v_1 < 0$, then the ω-limit of any orbit γ is $\omega(\gamma) \subset \mathbb{S}^2$ and the α-limit is

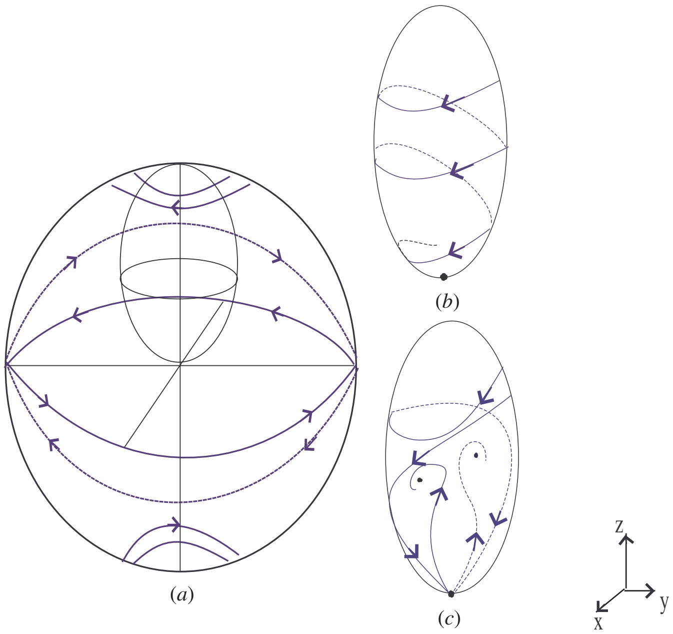

Figure 18. Phase portrait of the Rabinovich system for $v_1 = v_2 = 1$, $v_3 = 2$, $h > 0$ on the sphere in (*a*), and on the paraboloid $x^2 + y^2 - 4hz = 0$ with $0 < h \leqslant 1$ in (*b*) and with $h > 1$ in (*c*).

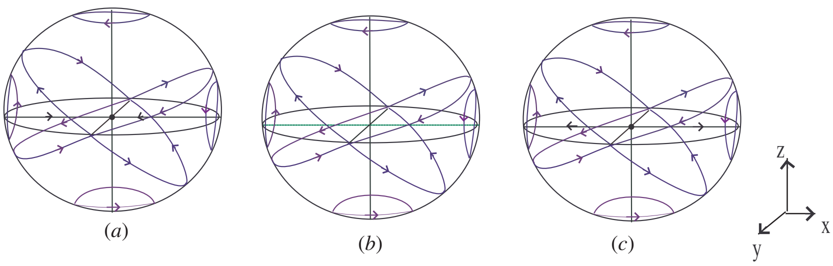

Figure 19. Phase portrait of the Rabinovich system for $v_2 = v_3 = v \neq 0$, $v_1 \neq v_2$, $h = 0$ on the sphere with $v_1 > 0$ in (*a*), with $v_1 = 0$ in (*b*), and with $v_1 < 0$ in (*c*).

contained on $\mathbb{S}^2$ too. If $v < 0$ and $v_1 = 0$, then the ω-limit of any orbit γ is $\omega(\gamma) \subset x$-axis. See figure 19.

4.10. Case $v_1 = v_3 = v \neq 0$, $v_2 \neq v$, $h = 0$ (table 1(j))

The planes $x = \pm z$ are invariant by the flow of system (1). A complete analysis of the system restricted to the levels $x = \pm z$ can be found in [3]. Taking $x = z$ the Rabinovich system reduces to

$$\dot{x} = -v_1 x + xy, \qquad \dot{y} = -v_2 y - x^2. \tag{15}$$

If $v_2 = 0$, then system (15) has a line of equilibria on the y-axis. The linear part of the system calculated at the equilibria $(0, y)$ has the trivial eigenvalues 0 and $y - v_1$,

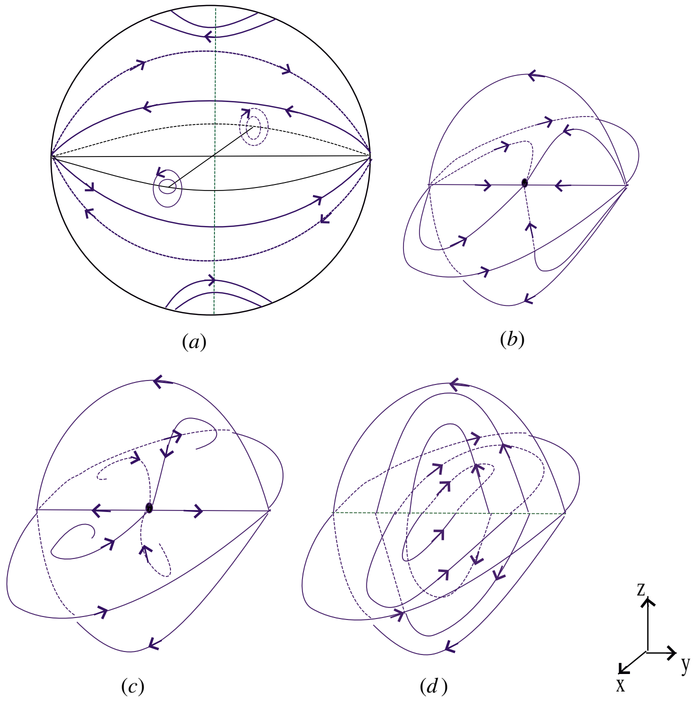

Figure 20. Phase portrait of the Rabinovich system for $v_1 = v_3 = v \neq 0$, $v_2 \neq v$, $h = 0$ on the sphere at infinity in (*a*), with $v_2 > 0$ in (*b*), with $v_2 < 0$ in (*c*), and with $v_2 = 0$ in (*d*).

with corresponding eigenvectors (0, 1) and (1, 0), respectively. Now by a simple integration we have that the orbits of system (15) are contained in the ellipses given by the equation $\frac{x^2}{2} + \frac{y^2}{2} - v_1 y = c$, with $c > 0$ constant. The endpoints of the invariant planes $x = \pm z$ coincide exactly with the heteroclinic cycles of the saddles at infinity. The analysis in the plane $x = -z$ is completely analogous.

If $v_2 \neq 0$, the unique equilibrium point of system (15) is the origin (0, 0). If $v_1 = v_3 > 0$ and $v_2 > 0$ the origin is the unique equilibrium point and it is globally asymptotically stable. If $v_2 < 0$ the origin is a saddle point. On the direction of the y-axis, which is invariant under the flow of the system, the origin is asymptotically stable if $v_2 > 0$ and unstable if $v_2 < 0$.

According to lemma 5 the ω-limit set of any point not in the planes $x = \pm z$ are contained in these planes and the α-limit set is at infinity.

If $v_1 = v_3 > 0$ and $v_2 < 0$ the system has four more equilibria beyond the origin, two of them contained in the plane $x = z$ and the other two in the plane $x = -z$. These equilibria are unstable foci (see figures 20 and 21).

5. Proof of theorem 4 and guided numerical simulations

We start this section with the proof of theorem 4.

Proof of theorem 4. For the parameter values $h = v_2 = 0$ and for any $v_1 = v_3 \neq 0$, system (1) has a line of equilibria, given by the y-axis, see figure 20(*d*). For $y < v_1$ the equilibria $P = (0, y, 0)$ are stable in a normal direction to the y-axis, that is the

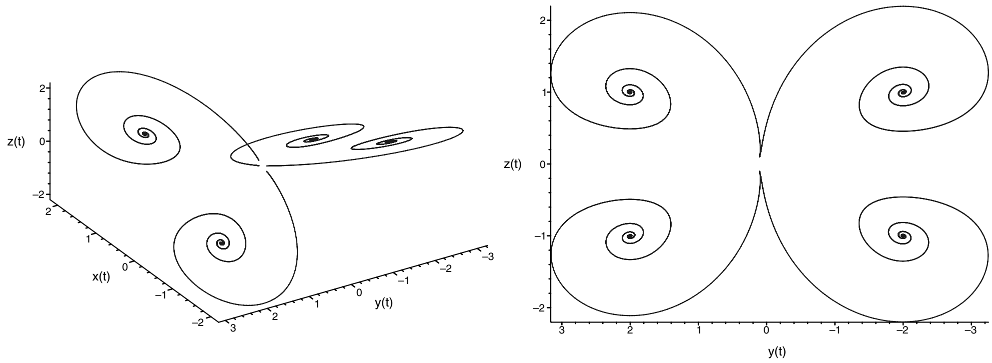

Figure 21. Phase portrait of the Rabinovich system with $h = 0$, $v_1 = v_3 > 0$ and $v_2 < 0$ and its projection in the plane yz.

linearized system at P has one negative and two zero eigenvalues and the corresponding one-dimensional stable manifolds are contained in the planes $x = \pm z$ and consequently are normal to the y-axis. For $y > v_1$ the equilibria $Q = (0, y, 0)$ are unstable in a normal direction to the y-axis, that is the linear part of system (1) at Q has one positive and two zero eigenvalues and the corresponding one-dimensional unstable manifolds are contained in the planes $x = \pm z$, and hence are normal to the y-axis. The one-dimensional unstable manifolds $W^u(Q)$ of each normally hyperbolic equilibrium Q tend to one of the normally hyperbolic stable equilibrium P as $t \to +\infty$, forming singularly degenerate heteroclinic cycles (see figure 20(*d*)). There is one family of degenerate heteroclinic cycles contained in the plane $x = z$ and another contained in the plane $x = -z$. Furthermore, when we tend to infinity these cycles accumulate in the heteroclinic cycles connecting the saddles on the sphere at infinity (see figure 2). □

Now, guided by the results of theorem 2 and by the suggestions of [10], we present some numerical simulations which indicate that system (1) may present strange attractors in a neighborhood of the families of singularly degenerate heteroclinic cycles.

Taking into account the results presented in this and in the previous sections one can conclude that it is impossible to encounter any type of strange attractor near the singularly degenerate heteroclinic cycles only by taking $v_2 \neq 0$. Then we have to use the parameter h and consider $v_1 \neq v_3$. Following these guidelines provided by the analytical results, we numerically found a four wings butterfly shaped strange attractor for system (1) with parameter values $v_1 = 1.5$, $v_2 = -0.3$, $v_3 = 1.67$ and $h = 0.04$. This attractor and its projections in the coordinate planes are shown in figures 4 and 5 of section 1.

One of the main properties of chaotic systems is their sensitive dependence on initial conditions. In order to check numerically if this property is verified for the Rabinovich four wings attractor we have made several numerical simulations. Considering, for instance, the initial conditions $(5, 5, 5)$ and $(5, 5.01, 5)$ we got the three-dimensional attractor and respective x-coordinate, the parametrized curve $(t, x(t))$, as shown in figure 22. The solutions related to each initial condition differ in its gray scale. One can observe clearly that the solutions stay close up to the time value $t = 70$ and then they diverge drastically, showing the mentioned sensitive dependence on initial conditions.

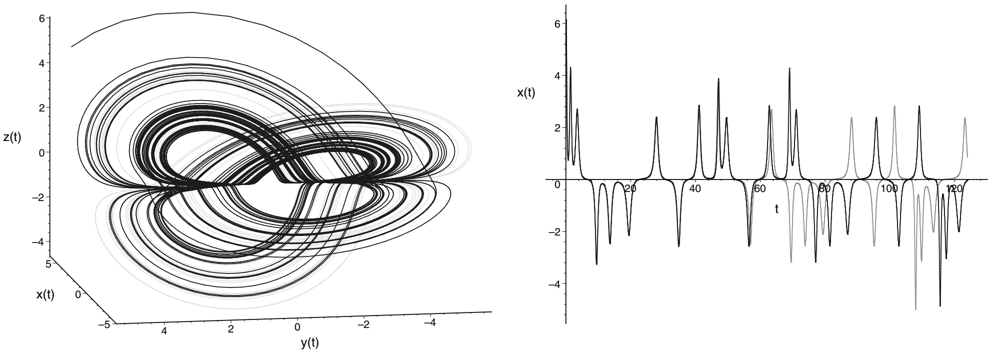

Figure 22. Strange attractor of the Rabinovich system with $v_1 = 1.5$, $v_2 = -0.3$, $v_3 = 1.67$ and $h = 0.04$ (left) and its x-coordinate curve (right). Initial conditions: (5,5,5) and (5,5.01,5).

5.1. Remarks on the paper [3]

We have some remarks about the paper [3].

(a) In the abstract the authors present the Rabinovich system with the third equation given by $\dot{z} = v_3 z + xy$ although throughout the paper and also in the related papers the third equation is $\dot{z} = -v_3 z + xy$.

(b) In theorem 1.1(b) of [3] (see table 11(b)) it is stated that for $v_1 = v_2 = v_3 \neq 0$, and $h = 0$ the Rabinovich system is completely integrable and that $H_1 = (x^2 + y^2)/(x^2 - z^2)$ and $H = (y^2 + z^2)/(x^2 - z^2)$ are two first integrals. In fact these two functions are first integrals but unfortunately they are not independent.

(c) The singular point (0, 0, 0) corresponding to the Rabinovich system with $v_1 = v_2 = v_3 \neq 0$, $h \neq 0$ is an attracting node or focus if $|h| < 1$, and a saddle if $|h| > 1$. Observe that it is not correctly represented in figure 7 of [3].

Acknowledgments

The first author is partially supported by a MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550. The second author is partially supported by CNPq-Brazil under the project 478544/2007-3. The third author is partially supported by CAPES, CNPq and FAPESP.

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